$(4-x)(4+x)$ $\frac{\sin x}{7}$ $1+\cos x$

$4-x$

PART I: WRITE THE MOST SIMPLIFIED ANSWER ON THE SPACE PROVIDED (1.5 each blank space)

1) Compute the following limits

   a) $\lim_{x\to 4^-} \frac{16-x^2}{|x-4|}=$ 8

   b) $\lim_{x\to 0} \frac{\sin^3 x}{x - x\cos x}=$ 2

2) If $f(x) = \int_1^{x^2} \sinh(t+1)dt$,then $f'(x) =$____________

3) Compute the following integrals

   a) $\int \sin(5x)\cos(3x)dx=$____________

   b) $\int_{-\infty}^{\infty} \frac{4}{3x^2+3} dx=$____________

4) Given $g(t) = f(h(t))$, $h(1) = 4$, $f'(4) = 3$ and $h'(1) = -6$ then $g'(1) =$ -18

5) The value(s) of a and b such that a function $f(x) = \frac{ax-b}{x^2+1}$ has local extreme value of 3 at $x = 1$. $a =$ 6 $b =$ 0

6) Equation of the tangent line to the curve $x^2y + xy^2 = 3x$ at the point $(2, -3)$ is

   $y = x-5$

7) If $f(x) = \begin{cases} \frac{4\sin(kx)}{x} & if\ x < 0 \\ x^2 - k^2 & if\ x \geq 0 \end{cases}$ is continuous at $x = 0$,then $k =$ $-4, 0$

8) Given $f(x) = (\ln x)^{\cos x}$,then $f'(x) =$ $(\ln x)^{\cos x}\left(-\sin x \ln(\ln x) + \frac{\cos x}{x\ln x}\right)$

PART II: WORK OUT; SHOW ALL THE NECESSARY STEPS CLEARLY AND NEATLY.

1) Using an $\varepsilon - \delta$definition of limit show that $\lim_{x\to 1} 2x^2 + \frac{3}{2}x + 1 = \frac{9}{2}$

for $\forall \varepsilon$, $\exists \delta$ such that

$0 < |x-1| < \delta$, then $|2x^2 + \frac{3}{2}x + 1 - \frac{9}{2}| < \varepsilon$

but $|2x^2 + \frac{3}{2}x - \frac{7}{2}| < \varepsilon$

$2|x^2 + \frac{3}{4}x - \frac{7}{4}| < \varepsilon$

$2|(x-1)(x+\frac{7}{4})| < \varepsilon$ $\quad 2|(x-1)(x+\frac{7}{4})| < \varepsilon$

Let $\delta_1 = 1$ $\quad |x-1| < \delta_1$ $\quad -1 < x-1 < 1$

$0 < x < 2$

$2|(x-1)||\frac{15}{4}| < \varepsilon$ $\quad \frac{7}{4} < |x+\frac{7}{4}| < \frac{15}{4}$

$|x-1| < \frac{2}{15}\varepsilon$ $\quad \delta < \frac{2}{15}\varepsilon$

We can choose $\delta = (1, \frac{2}{15}\varepsilon)$

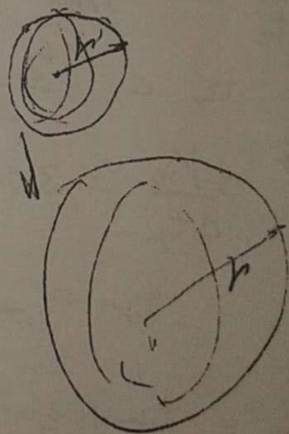
2) A spherical balloon is expanding. Given that the radius is increasing at the rate of 2 meters per minute, at what rate is the volume increasing when the radius is 5 meters.

$V = \frac{4}{3}\pi r^3$

$r = 5$ m

$\frac{dr}{dt} = 2$ m/min

what is $\frac{dV}{dt} = ?$



$\frac{dV}{dt} = \frac{4}{3}\pi 3r^2 \frac{dr}{dt}$

$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3(5)^2 \cdot 2$

$\frac{dV}{dt} = 200\pi$ meter^3/minute

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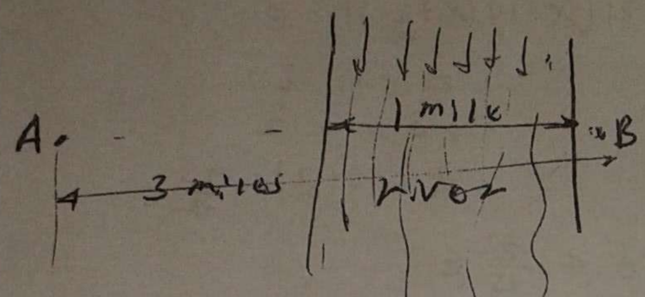
3) A river is 1 mile wide. Abera wants to get from point A to point B on the opposite side of the river, 3 miles down stream. If Abera can run 5 miles per hour and can swim 3 miles per hour, what is the least amount of time in which he can get from A to B ?



It run on Land and swim on river

on Land 2 miles Land b/c 3 mile − 1 mile = 2 miles

V_L = speed = 5 mile/hr

$V_L = \frac{\text{wide of Land}}{t_L}$

$\frac{dLw}{dt_L} = 5$ mile/hr $= \frac{2 \text{ mile}}{t_L}$

$t_L = \frac{2 \text{ mile}}{5 \text{ mile/hr}} = \frac{2}{5}$ hr

$\frac{drw}{dt_r} = 3$ mile/hr $= \frac{\text{wide of river}}{t_r}$

$t_r = \frac{1 \text{ mile}}{3 \text{ mile/hr}} = \frac{1}{3}$ hr

$t_{total} = t_r + t_L = \frac{2}{5}$ hr $+ \frac{1}{3}$ hr $= \frac{6+5}{15}$ hr $= \frac{11}{15}$ hr

(20)

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4) Evaluate the following integrals

a) $\int \frac{x^2}{\sqrt{16-x^6}}\,dx$

b) $\int \frac{2x}{x^3+x^2+x+1}\,dx$

(b)

$$\int \frac{2x}{x^3+x^2+x+1}\,dx = \int \frac{2x}{x^2(x+1)+1(x+1)}\,dx$$

$$\int \frac{2x}{(x^2+1)(x+1)}\,dx = \frac{Ax+B}{x^2+1} + \frac{C}{x+1}$$

$$Ax^2+Bx+Ax+B+Cx^2+C = 2x$$

$$A+C=0 \qquad A=-C$$

$$B+A=2$$

$$B+C=0 \qquad B=-C$$

$$\Rightarrow A=B$$

$$2A=2 \qquad A=1 \qquad B=1$$

$$C=-1$$

$$\int \frac{2x}{(x^2+1)(x+1)}\,dx = \int \frac{2x}{x^3+x^2+x+1}\,dx = \int \frac{x+1}{x^2+1}\,dx - \int \frac{1}{x+1}\,dx$$

$$\int \frac{x+1}{x^2+1}\,dx = \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$$

let $v=x^2+1$ $\quad dv=2x\,dx \quad x\,dx=\frac{dv}{2}$

$$\int \frac{x}{x^2+1}\,dx = \int \frac{1}{v}\frac{dv}{2} = \frac{1}{2}\ln v = \frac{1}{2}\ln|x^2+1|$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}(x)$$

$$\int \frac{1}{x+1} = \ln|x+1|$$

$$\therefore \int \frac{2x}{x^3+x^2+x+1}\,dx = \frac{1}{2}\ln|x^2+1| + \tan^{-1}(x) - \ln|x+1| + C$$

5) Let $f(x) = \frac{x^2}{x^2-4x+4}$ then

a) Find the asymptote(s)
b) Find the interval on which $f(x)$ is increasing and decreasing
c) Find the local extremes of $f(x)$
d) Determine concavity and inflection point(s) $f(x)$
e) Sketch the graph of $f(x)$

a Assimptote

HA = ~~Hom~~ horizontal Assimptote
V.A = Vertical Assimptote

V.A $f(x) = \frac{x^2}{(x-2)^2}$

$\lim_{x \to a} f(x) = \pm\infty$ $a = 2$

$\therefore$ V.A is $x = 2$

HA, $\lim_{x \to \infty} f(x) = a$ $\lim_{x \to \infty} \frac{x^2}{x^2-4x+4} = 1$

$\therefore$ H.A $y = 1$

b Find interval on which f(x) is increasing and decreasing

$f(x) = \frac{x^2}{x^2-4x+4}$

$f'(x) = \frac{2x(x^2-4x+4) - x^2(2x-4)}{(x^2-4x+4)^2}$

$f'(x) = \frac{\cancel{2x^3} - 8x^2 + 8x - \cancel{2x^3} + 4x^2}{(x^2-4x+4)^2}$

$f'(x) = \frac{-4x^2+8x}{(x^2-4x+4)^2}$

61

3rd find inflection point

$f''(x) = 0$

$$\frac{-2x+14}{(x-1)^4} = 0$$

$-2x+14 = 0$

$2x = 14$

$x = 7$ is inflection point

by using sign chart

$-2x+14 = 14-2x$

	$(-\infty, 1)$	1	$(1, 7)$	7	$(7, \infty)$
$(x-1)^4$	$+++$	0	$+++$	$+$	$++$
$14-2x$	$++$	$+$	$+++$	0	$----$
$\frac{14-2x}{(x-1)^4}$	$+++$	$\nexists$	$+++$	0	$----$

C) $\Rightarrow (-\infty, 1)$ and $(1, 7] \to$ concave upward

$[7, \infty) \to$ concave downward.

D) inflection point

at $x = 7$

$$f(x) = \frac{2+x-x^2}{(x-1)^2}$$

$$f(7) = \frac{2+7-49}{(7-1)^2} = \frac{-40}{36} = \frac{-10}{9}$$

$\left(7, \frac{-10}{9}\right)$ is inflection point

II

Adama Science and Technology University

School of Natural Sciences

Department of Mathematics

Applied Mathematics I (Math 131) Final Exam

Time allowed 2:30hrs.

(Personal Information)	(For Instructors Use Only)
Name Jola Bekele	
ID.No Alwe 443?/08	
Department: Pre-Engineering	
Group 40	
Regular ☑ Add ☐	

Parts	Score
I	All
II	All
III	All
Total	3 All

Part One: Multiple Choices

Direction: This session has 6 questions. Choose the best answer (2pts each)

D 1. Let $f(x) = e^3$ then $f'(x)$ is

a) e^3 b) $3e^2$ c) $\frac{1}{3}e^2$ d) 0

A 2. For what values of c the function $f(x) = \begin{cases} 3x + c, & if\ x < 2 \\ cx^2 - 2x + 4, & if\ x \geq 2 \end{cases}$ is continuous at $x = 2$

a) 2 b) 0 c) -2 d) no answer

B 3. The slope of the tangent line to the graph of a function $f(x) = 2x + 4\sqrt{x}$ at $x = 4$ is

a) 16 b) 3 c) 4 d) does not exist

E 4. If $\lim_{x\to a} f(x) = L$, then which of the following is true

a) $\lim_{x\to a^+} f(x) = L$

b) $\lim_{x\to a^-} f(x) = L$

c) $f(a) = L$

d) all

e) a & b

C 5. If $f(x) = 5^x$, then $f'(x)$ is

a) 0 b) $x5^{x-1}$ c) $5^x ln5$ d) $\frac{1}{5^x ln5}$

D 6. If $x^2y - 3x = y^3 - 3$, then at the point $(-1,2)$, $\frac{dy}{dx} =$

a) $-\frac{1}{7}$ b) $-\frac{7}{13}$ c) $-\frac{1}{2}$ d) $-\frac{7}{11}$ e) 7

8

Part II: Fill in the Blank Space

Direction: This part contains 7 questions. Write the most simplified answer on the space provided. (Each blank space worth 2 pts)

1. Given if $g(t) = f(h(t))$, $h(1) = 4$, $f'(4) = 3$ and $h'(1) = -6$, then $g'(1) =$ __-18__
2. Find the following limits
   a. $\lim_{x\to\infty} \frac{\sqrt{x}+3}{3-2x} =$ __0__
   b. $\lim_{x\to 1} \frac{\sin(x-1)}{x^2-1} =$ __$1/2$__
   c. $\lim_{x\to 4^-} \frac{16-x^2}{|x-4|} =$ __8__
3. Given that $f(x) = \frac{3x^2+4}{2-x^2}$ then
   a. The vertical asymptote of the graph of f is the line __$x = \pm\sqrt{2}$__
   b. The horizontal asymptote of the graph of f is the line __$y = -3$__
4. An equation of the tangent line to the graph of $f(x) = x\ln x$ at the point $(1,0)$ is __$y = x-1$__
5. If $f(x) = x^x$, then $f'(x) =$ __$x^x(\ln x+1)$__
6. Every continuous function is differentiable. (true/false) __FALSE__
7. Given that $f(x) = f(-x)$ and f is differentiable function, then $f'(0) =$ __0__
8. If the graph of the function $f(x) = \frac{e^{ax}}{a(x+1)^3}$ has a horizontal tangent line at 0, then $a =$ __3__

$g'(t) = f'(h(t))\,h'(t)$

$g'(1) = f'(h(1))\,h'(1)$

$= f'(4)(-6)$

$= 3(-6) = -18$

$\frac{(4-x)(4+x)}{-x+4}$

$4+x = 8$

$f'(x) =$

$f'(x) = \frac{a e^{ax}(x+1)^3 - 3a e^{ax}(x+1)^2}{a^2(x+1)^6}$

$= \frac{a e^{ax}(x+1) - 3a e^{ax}}{(x+1)^4} = 0$

$\Rightarrow f'(0) = 0$

$\frac{a-3}{1} = 0$

$a = 3$

Part III: Work Out Problems.

Direction: This part has 4 questions. Attempt all questions neatly. Unreadable answers may worth no point. (5pts each)

1. Use $\varepsilon-\delta$ definition to show that $\lim_{x\to 1} x^2 = 1$

for ever ε there exist some δ ($\forall\varepsilon, \exists\delta$)

~~These~~ (such that

$0<|x-1|<\delta$ then $|x^2-1|<\varepsilon$

$\Rightarrow |x-1||x+1| < \varepsilon$

Let $\delta_1 = 1$ $\quad |x-1|<1 \quad -1<x-1<1$

$0<x<2$

$1<|x+1|<2$

$\Rightarrow |x-1|(2) < \varepsilon$

$|x-1| < \frac{\varepsilon}{2}$

Chouse $\delta = (1, \frac{\varepsilon}{2})$

$\therefore \lim_{x\to 1} x^2 = 1$

2. Find the value of k so that the function $f(x) = \begin{cases} \frac{4\sin kx}{x} & if\ x<0 \\ x^2-k^2 & if\ x\geq 0 \end{cases}$ is continuous at $x = 0$

To find the value of k

evaluet $\lim_{x\to 0} \frac{4\sin kx}{x} = 4k$

$f(0) = 0^2 - k^2 = -k^2$

$f(0) = \lim \frac{4\sin kx}{x}$

$-k^2 = 4k$

$4k + k^2 = 0$

$k(k+4) = 0$

$k = 0$

$k+4 = 0$

$k = -4$

$\therefore$ The value of $k = 0$ and $k = -4$

3

10

3. Use definition of derivatives to find the derivative of $f(x) = \log_2 x$

~~Solution~~

Sol$^{\underline{n}}$

$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ is derivative by using def$\underline{n}$

$f(x) = \log_2 x \quad f(x+h) = \log_2 (x+h)$

$\lim_{h \to 0} \frac{\log_2 (x+h) - \log_2 x}{h}$

$\lim_{h \to 0} \frac{\log_2 \frac{x+h}{x}}{h}$ since $\log (x+h) - \log x = \log \frac{x+h}{x}$

$\lim_{h \to 0} \frac{1}{h} \log_2 \left(1 + \frac{h}{x}\right)$ - by log or lim Law

$\lim_{h \to 0} \log_2 \left(1 + \frac{h}{x}\right)^{\frac{1}{h}}$

$\log_2 \lim_{h \to 0} \left(1 + \frac{h}{x}\right)^{\frac{1}{h}}$

$\log_2 e^{\frac{1}{x}}$

$\frac{1}{x} \log_2 e$

$\frac{1}{x \log_e 2}$

$\frac{1}{x \ln 2}$

4. Compute the following limits

a. $\lim_{x\to 0}\frac{\cos^2(3x)-1}{x^2}$

$\cos^2 3x = 1-\sin^2 3x$

$\lim_{x\to 0}\frac{\cos^2 3x - 1}{x^2} = \lim_{x\to 0}\frac{\cancel{1}-\sin^2 3x-\cancel{1}}{x^2}$

$\lim_{x\to 0}\frac{-\sin^2 3x}{x^2} = -\left(\lim_{x\to 0}\frac{\sin 3x}{x}\right)^2$

$= -3^2$

$= -9$

or by using L'hopital rul

$\lim_{x\to 0}\frac{(\cos^2 3x-1)'}{(x^2)'}$

$\lim_{x\to 0}\frac{-3\sin 3x \cdot \cancel{2}\cos 3x}{\cancel{2}x}$

$= -3\lim \frac{\sin 3x}{x}$

-3×3

$= -9$

b. $\lim_{x\to 0}\frac{\sin^3 x}{x - x\cos x}$

$\lim_{x\to 0}\frac{\sin x \cdot \sin^2 x}{x(1-\cos x)}$

$\lim_{x\to 0}\frac{\sin x}{x}\left(\frac{1-\cos^2 x}{1-\cos x}\right)$

$\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\cancel{(1-\cos x)}(1+\cos x)}{\cancel{(1-\cos x)}}$

$\lim_{x\to 0}\frac{\sin x}{x}\quad \lim_{x\to 0} 1+\cos x$

$1 \cdot 2$

$= 2$

~~by using L'hopital rule~~

$\lim_{x\to 0}\frac{(\sin^3 x)'}{(\cancel{x - x\cos x})'}$ $\quad \cancel{\frac{-3\sin^2 x\cos x}{2-\cos x}}$ $\quad \cancel{\frac{6\sin^3 x}{\cos x}}$

$\frac{3\sin^2 x\cos x}{1E(\cos x + x\sin x)}$ $\quad \cancel{(1-\cos x)}(1+\cos x)$

$\cancel{\frac{3\cdot 2\sin x\cos^2 x - 3\sin^3 x}{\sin x + \sin x + x\cos x}}$

$\frac{\cancel{6\sin x - 6\sin^3 x}}{2\sin x + x\cos x}$

12

PART I: SHORT -ANSWER QUESTIONS

Read carefully and give the answer in its most simplified form

1. $\lim_{x\to0} x\sin\left(\frac{1}{x}\right) =$ ____ 0 ____

2. $\lim_{x\to1^-} \frac{\sin\sqrt{1-x}}{\sqrt{1-x^2}} =$ ____ $\frac{1}{\sqrt{2}}$ ____

3. If $\lim_{x\to0} \frac{f(x)}{x^2} = -2, then\ \lim_{x\to0}\left(\frac{f(x)}{x}+2\right)$= ____ 2 ____.

4. $\lim_{x\to+\infty}(x-\sqrt{x^2-3x}) =$ ____ $\frac{3}{2}$ 0 ____

5. $\lim_{x\to1}(2-x)^{\frac{1}{1-x}} =$ ____ e^{-1} e^{1} ____

6. If $f(x) = \begin{cases} ax-b & if \quad x\le -1 \\ 2x^2+3ax+b & if \quad -1<x\le 1 \\ 4 & if \quad x>1 \end{cases}$ is continuous for all x, then

the constants a = ____ 1 $\frac{3}{-4}$ ____, b= ____ $\frac{1}{3}$ $-\frac{1}{4}$ ____

7. Give an example of a function which is continuous at a point but not differentiable at that point? Specify that point ? Ans ____ $f(x) = |x|$ at $x = 0$ ____.

8 . If $f(x)$ and $g(x)$ are differentiable functions such that $f'(x) = 3x\ and\ g'(x) = 2x^2$,

$then \quad \lim_{x\to1}\frac{[f(x)+g(x)]-[f(1)+g(1)]}{x-1} =$ ____ 5 ____.

9. The n^{th} derivative of $f(x) = a^{2x}$. ____ $f^n(x) = (2\ln a)^n\ a^{2x}$ ____

10. let $f(x) = x^{\ln(\sin x)}$ Then $f'(\frac{\pi}{2})$= ____ 0 ____.

14

11. The equation of normal line to the graph of $f(x) = \frac{1}{\sqrt{x}}$ at the point (1,1)is $y = 2x - 1$

12. Find a number c that satisfies the mean value theorem for the

function$f(x) = \frac{x}{1+x}$ on$[0,3]$ .Ans $c = 1$

13. The maximum value of $f(x) = cosx + sinx$ on $[0, \pi] is$ $\sqrt{2}$.

14. If $\int_0^1 \sqrt{1-x^2}\, dx = \frac{\pi}{4}, then \int_0^1 4(\sqrt{1-x^2} + 2)dx =$ $\pi + 8$.

15. Let $F(x) = \int_0^{2x^3} \frac{1}{\sqrt{t^2+1}} dt$, then F'(x)=____________________.

16. $\int_{-2}^{2} |x-1|\, dx =$ ______________________________.

15

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11. The equation of normal line to the graph of $f(x) = \frac{1}{\sqrt{x}}$ at the point (1,1)is $y = 2x - 1$

12. Find a number c that satisfies the mean value theorem for the

function $f(x) = \frac{x}{1+x}$ on $[0,3]$ .Ans $c = 1$

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14. If $\int_0^1 \sqrt{1-x^2}\, dx = \frac{\pi}{4}$, $then$ $\int_0^1 4(\sqrt{1-x^2} + 2)dx =$ $\pi + 8$.

15. Let $F(x) = \int_0^{2x^3} \frac{1}{\sqrt{t^2+1}} dt$, then $F'(x)=$________________.

16. $\int_{-2}^{2} |x-1|\, dx =$ ________________.

15

PARTII: WORK - OUT PROBLEMS.

Solve each problem in this part in detail giving the necessary justification in the space provided.

1 Using **only** the definition of derivatives, find $f'(x)$ where $f(x) = \frac{1}{\sqrt{x+1}}$. **(5 points)**

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\lim_{h \to 0} \frac{\frac{1}{\sqrt{x+h+1}} - \frac{1}{\sqrt{x+1}}}{h}$$

$$\lim_{h \to 0} \frac{\sqrt{x+1} - \sqrt{x+h+1}}{h\sqrt{(x+1)(x+h+1)}} \cdot \frac{(\sqrt{x+1} + \sqrt{x+h+1})}{\sqrt{x+1} + \sqrt{x+h+1}}$$ rationalize

$$\lim_{h \to 0} \frac{x+1 - (x+h+1)}{h\sqrt{(x+1)}\sqrt{x+h+1}} \cdot \frac{1}{\sqrt{x+1} + \sqrt{x+h+1}}$$

$$\lim_{h \to 0} \frac{-h}{h} \cdot \frac{1}{\sqrt{x+1}} \cdot \frac{1}{\sqrt{x+h+1}} \quad \frac{1}{\sqrt{x+1} + \sqrt{x+h+1}}$$

$$\frac{-1}{\sqrt{x+1}} \cdot \frac{1}{\sqrt{x+0+1}} \quad \frac{1}{\sqrt{x+1} + \sqrt{x+0+1}}$$

$$\frac{-1}{x+1} \cdot \frac{1}{2\sqrt{x+1}}$$

$$\frac{-1}{2(x+1)\sqrt{x+1}}$$

16

2. Find the value(s) of m and b such that the function $f(x) = \begin{cases} x^2 - 3x + 2, & x \le 1 \\ mx + b, & x > 1 \end{cases}$

is differentiable for all values of x.

(5 points)

$f(1) = 1^2 - 3(1) + 2 = 0$

$\lim_{x \to 1^+} mx + b = m + b$

$f(1) = \lim_{x \to 1^+}$

$0 = m + b$

$m = -b$

for every value $m = -b$

3. The curve $y = ax^3 + bx^2 + cx + d$ has horizontal tangent lines at the points (0, 1) and (1, 0). Then find the constant s a, ,b, c and d. **(5 points)**

Soln

slope of horizontal tangent lines = 0

It pass on the point (0, 1) & (1, 0)

on (0, 1) $1 = a(0)^3 + b(0)^2 + c(0) + d$

$d = 1$ --- eqn (1)

$0 = a(1)^3 + b(1)^2 + c(1) + d$

$a + b + c + d = 0$ --- eqn (2)

$y' = f'(x) = \text{slope} = 3ax^2 + 2bx + c$

on point (0, 1)

$f'(0) = 3a(0)^2 + 2b(0) + c = c = 0$

$c = 0$ --- eqn (3)

on point (1, 0)

$f'(1) = 3a(1)^2 + 2b(1) + c = 0$

$3a + 2b + c = 0$ --- eqn (4)

in eqn (2) $a + b + c + d = 0$ $d = 1$, $c = 0$

$a + b + 0 + 1 = 0$

$a + b = -1$

in eqn (3)

$3a + 2b + c = 0$

$3a + 2b = 0$

$b = -\frac{3}{2}a$

$a - \frac{3}{2}a = -1$

$-\frac{a}{2} = -1$

$a = 2$

$b = -a - 1$

$b = -2 - 1$

$b = -3$

18

-5-

$\therefore$ $\boxed{a = 2,\ b = -3,\ c = 0,\ d = 1}$ → Answer

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4. Let $f(x) = \frac{2+x-x^2}{(x-1)^2}$. Then find a) the interval on which f is increasing? , decreasing?

b) local extreme point(s)? c) the interval on which f is concave up?, concave down?

d) inflection point(s)? **(7 points)**

Solution

$$f(x) = \frac{2+x-x^2}{(x-1)^2}$$

$$f'(x) = \frac{(1-2x)(x-1)^2 - (2+x-x^2)\,2(x-1)}{((x-1)^2)^2}$$

$$f'(x) = \frac{(1-2x)(x-1) - (2+x-x^2)\,2}{(x-1)^3}$$

$$\frac{(x-2x^2-1+2x) - 4 - 2x + 2x^2}{(x-1)^3}$$

$$\frac{x-5}{(x-1)^3} = 0$$

find the critical point $x-5=0$ $x=5$ ~~is the~~ is the only critical point since $x=1$ is not domain not critical point

by using sign chart method

	1	5	
$(x-1)^3$	- - - - - - 0	+ + + + + +	+ + + + +
$(x-5)$	- - -	- - - - - - 0	+ + + + +
$\frac{x-5}{(x-1)^3}$	+ + + + + + ∅	- - - - - - 0	+ + + + + + +

a $\Rightarrow (-\infty, 1)$ and $[5, \infty)$ f is increasing

$\Rightarrow (1, 5]$ - f is decreasing

19

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(b) local ~~extreme~~ extreme point

at $x = 5$

$$f(x) = \frac{2+x-x^2}{(x-1)^2}$$

at $x = 5$

$$f(x) = \frac{2+5-5^2}{(5-1)^2} = \frac{7-25}{4^2} = -\frac{18}{16} = -\frac{9}{8}$$

$\therefore \left(5, -\frac{9}{8}\right)$ local extreme point

c To find concavity

1st find $f''(x)$

$$f''(x) = \left(\frac{x-5}{(x-1)^3}\right)'$$

$$f''(x) = \frac{(x-1)^3 - (x-5)\,3(x-1)^2}{(x-1)^6}$$

$$f''(x) = \frac{(x-1) - (x-5)\cdot 3}{(x-1)^4}$$

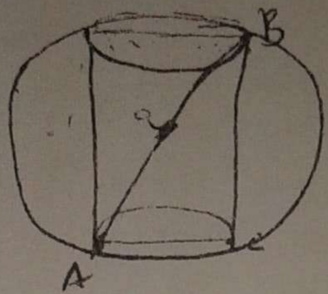
$$f''(x) = \frac{(x-1) - 3x + 15}{(x-1)^4}$$

$$f''(x) = \frac{x-3x-1+15}{(x-1)^4}$$

$$f''(x) = \frac{-2x+14}{(x-1)^4}$$

5. What is the largest possible volume of right circular cylinder that can be inscribed in a sphere of radius 5 units?

(6 points)



AB is ~~diagonal~~ diameter of sphere

AC diameter of cylinder

BC height of cylinder

R = radius of sphere

r = radius of cylinder

$$AB^2 = AC^2 + BC^2$$

$$(2R)^2 = (2r)^2 + h^2$$

$$\boxed{4R^2 = 4r^2 + h^2}$$

$$4r^2 = 4R^2 - h^2$$

$$r^2 = R^2 - \frac{h^2}{4}$$

$$V_{of\ cylinder} = \pi r^2 h$$

$$V_{cylinder} = \pi \left(R^2 - \frac{h^2}{4}\right) h$$

$$V_{cylinder} = \pi R^2 h - \frac{h^3}{4}$$

when $R = 5$

$R^2 = 25$

$$V_{cylinder} = 25\pi h - \frac{h^3}{4}$$

$$V_{max} = \quad \frac{dV}{dh} = 25\pi - \frac{3h^2}{4} = 0$$

$$25\pi - \frac{3h^2}{4} = 0$$

$$h^2 = \frac{100\pi}{3}$$

$$h = \sqrt{\frac{100\pi}{3}} \text{ unit}$$

$$r^2 = 25 - \frac{\frac{100\pi}{3}}{4}$$

$$r^2 = \frac{300 - 100\pi}{12}$$

$$V_{max} = \pi r^2 h = \pi \cdot \frac{300 - 100\pi}{12} \cdot \sqrt{\frac{100\pi}{3}} \text{ unit cube}$$

$$= \frac{300\pi - 100\pi^2}{12} \sqrt{\frac{100\pi}{3}} \text{ unit cube}$$

21

= ________________ unit cube.

6. Evaluate the following integrals.

a) $\int e^{3x} \cos(3x) dx$ (4 points)

b) $\int \frac{\sqrt{x^4}}{x^2+4x+4} dx$ (4 points)

a $\int e^{3x} \cos(3x)\, dx$

Integration by part

$u = \cos 3x$

$du = -3\sin 3x\, dx$

$dv = e^{3x}\, dx$

$v = \frac{e^{3x}}{-3}$

$\int u\,dv = \frac{\cos 3x\, e^{3x}}{3} - \int \frac{e^{3x}}{\cancel{3}} \cdot -\cancel{3}\sin 3x\, dx$

$= \frac{\cos(3x) e^{3x}}{3} + \int e^{3x} \sin 3x\, dx$

by part

$dv = e^{3x} dx \quad v = \frac{e^{3x}}{3}$

$u = \sin 3x$

$du = 3\cos 3x\, dx$

$\int u\,dv = \frac{e^{3x}}{3}\sin 3x - \int \frac{e^{3x}}{\cancel{3}} \cancel{3}\cos 3x\, dx$

$\int e^{3x}\cos(3x)\, dx = \frac{\cos(3x) e^{3x}}{3} + \frac{e^{3x}\sin 3x}{3} - \int e^{3x}\cos 3x\, dx$

$2\int e^{3x}\cos 3x\, dx = \frac{\cos 3x\, e^{3x} + \sin 3x\, e^{3x}}{3}$

22 $\int e^{3x}\cos 3x\, dx = \frac{e^{3x}}{3}(\cos 3x + \sin 3x)$

$+C$

6. $\int \frac{x^4}{x^2+4x+4} dx$

$$\begin{array}{r} x^2 - 4x + 12 \\ x^2+4x+4 \overline{)\, x^4 + 0x^3 + 0x^2 + 0x + 0} \\ -x^4 + 4x^3 + 4x^2 \\ -4x^3 - 4x^2 + 0x + 0 \\ +4x^3 + 16x^2 + 16x \\ 12x^2 + 16x \\ -12x^2 + 48x + 48 \\ -32x - 48 \end{array}$$

$$\int \frac{x^4}{x^2+4x+4} dx = \int (x^2 - 4x + 12) dx - 16 \int \frac{2x+3}{(x+2)^2} dx$$

$$= \frac{x^3}{3} - \frac{4x^2}{2} + 12x - 16 \int \frac{2x+3}{(x+2)^2} dx$$

$\cancel{\frac{2x+3}{(x+2)^2}} = \frac{\cancel{A}}{\cancel{x+2}} + \frac{\cancel{B}}{\cancel{(x+2)^2}} = \cancel{4x^2 + 4xA + 4A + 8x + 28}$ $= \cancel{2x+3}$

$$\int \frac{2x+3}{(x+2)^2} dx = \int \frac{2x}{(x+2)^2} dx + \int \frac{3}{(x+2)^2} dx$$

23

PART I: Short answer (Each blank space worth 1.5 points)

Write the simplified answer on the space provided.

1. Find a formula for the inverse of each function

   a) $f(x)=\sqrt{1+x}$ $f^{-1}(x) = x^2 - 1$

   b) $g(x)=\dfrac{x+1}{x-1}$ $g^{-1}(x) = \dfrac{2}{x-1} + 1$

2. Differentiate the following functions

   a) $f(x)=\cos^{-1}\left(\sqrt{1+x^2}\right)$

   b) $g(x) = \sinh^{-1}(5x^2)$ $\dfrac{10x}{\sqrt{25x^4+1}} = \dfrac{10x}{\sqrt{25x^4+1}}$

   c) $h(x) = \tanh(\ln x)$ $\frac{1}{x}\,\mathrm{sech}(\ln x)\left(1+\tanh(\ln x)\right)$

3. Simplify each of the following

   a) $\sec(\arctan\sqrt{1-x})$

   b) $\tan\left(\csc^{-1}\dfrac{x}{2}\right)$

   c) $\tan(\arctan 10)$ 10

4Evaluate the following

   a) $\int x\sec^2 x\,dx$ $x\tan x + \ln|\cos x| + C$

   b) $\int \sin^3 x\cos^2 x\,dx$ $-\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C$

   c) $\int \dfrac{1}{x^2\sqrt{4+x^2}}\,dx$

   d) $\int \dfrac{e^x}{1+e^{2x}}\,dx$

$x^2 + 1 = c$

$x = \frac{x+1}{x-1}$

$\frac{2}{x-1} + 1$

PART II: Work out Problems

For the following problems show all the necessary steps clearly and legibly.

1. Let $f(x)=x^3+2x-5$, then find $(f^{-1})'(-2)$ (2 points)

2

soln $f(a) = c$

$f(a) = a^3+2a-5$

$-2 = a^3+2a-5$

$a^3+2a-5+2=0$

$a^3+2a-3=0$

$a=1$

$f'(x) = x^3+2x-5$

$= 3x^2+2$

$f'(-2) = \dfrac{1}{f'(a)} = \dfrac{1}{f'(3x^2+2)} = \dfrac{1}{3(1)^2+2} = \dfrac{1}{3+2} = \dfrac{1}{5}$

24

2. Evaluate the following integral (3 points each)

a) $\int \frac{x+1}{x^3+x^2-6x}\,dx$

b) $\int \tan^{3} x \sec^{4} x\,dx$

Sln

a $\int \frac{x+1}{x^3+x^2-6x}\,dx = \int \frac{x+1}{x(x^2+x-6)}\,dx$

$= \int \frac{x+1}{x(x+3)(x-2)}\,dx$

by partial fraction

$\frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-2} = \frac{x+1}{x(x+3)(x-2)}$

$A(x^2+x-6) + B(x^2-2x) + C(x^2+3x) = x+1$

$A+B+C=0 \quad \Rightarrow \quad B+C = \frac{1}{6}$

$A-2B+3C=1$

$-6A=1 \qquad A = -\frac{1}{6}$

$-2B+3C = 1+\frac{1}{6} = \frac{7}{6} \qquad 3C = 2B+\frac{7}{6}$

$B+C = \frac{1}{6} \Rightarrow B = \frac{1}{6} - C$

$3C = 2\left(\frac{1}{6}-C\right)+\frac{7}{6} \qquad 3C = \frac{1}{3} - 2C + \frac{7}{6}$

$5C = \frac{9}{6} = \qquad C = \frac{9}{30}$

$B = \frac{1}{6} - \frac{9}{30} = -\frac{4}{30}$

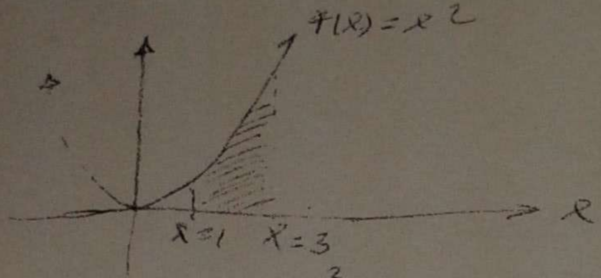
$\int \frac{x+1}{x^3+x^2-6x}\,dx = -\int \frac{1}{6x}\,dx - \frac{4}{30}\int \frac{1}{x+3}\,dx + \frac{9}{30}\int \frac{1}{x-2}\,dx$

$= -\frac{1}{6}\ln x - \frac{4}{30}\ln|x+3| + \frac{9}{30}\ln|x-2|$

$\int \frac{x+1}{x^3+x^2-6x}\,dx = \frac{9}{30}\ln|x-2| - \frac{4}{30}\ln|x+3| - \frac{1}{6}\ln|x| + C$

5

3. Find the area of the region R bounded by the graph of $f(x)=x^2$ and the line $x=1$, $x=3$ and the x-axis. (3 points)



$$Area = \int_1^3 (f(x) - 0)\, dx = \int_1^3 x^2 dx = \frac{x^3}{3}\Big|_1^3$$

$$Area = \frac{(3)^3}{3} - \frac{(1)^3}{3} = \frac{27}{3} - \frac{1}{3} = \frac{26}{3}$$

4. A radio active element has half-life of ln2 weeks. If e^3 kg present at a given time. How much will be left after 3 weeks? (3 points)

$$f(t) = ce^{kt}$$

$$f(0) = ce^{k(0)}$$

$$e^3 kg = ce^0$$

$$c = e^3$$

1.5

26

4

5. Find the volume of the solid obtained by revolving the region between the graph of $f(x) = 2x + 3$ and $g(x) = x^2$ about x-axis.(3 points)

$$V = \int_a^b \pi (f(x))^2 - (g(x))^2 \, dx$$

$$2x + 3 = x^2$$

2.

$$x^2 - 2x - 3 = 0$$

$$x = -1, \quad x = 3$$

$$V = \int_{-1}^{3} \pi (2x+3)^2 - [x^2]^2 \, dx$$

$$V = \int_{-1}^{3} \pi (4x^2 + 12x + 9 - x^4) \, dx$$

$$\pi \int_{-1}^{3} 4x^2 + 12x + 9 - x^4 \, dx$$

$$\pi \left(\frac{4x^3}{3} + \frac{12x^2}{2} + 9x - \frac{x^5}{5} \right) \Big|_{-1}^{3}$$

$$\pi \left[\left(\frac{4}{3}(3)^3 + \frac{12(3)^2}{2} + 9(3) - \frac{3^5}{5} \right) - \left(\frac{4}{3}(-1)^3 + \frac{12(-1)^2}{2} + 9(-1) - \frac{(-1)^5}{5} \right) \right]$$

$$\pi \left(\left(36 + 54 + 27 - \frac{243}{5} \right) - \left(\frac{-4}{3} + 6 - 9 + \frac{1}{5} \right) \right)$$

$$\cancel{\pi(117 - 48.6)} - \left(\frac{-20 + 3 - 45}{15} \right.$$

$$\pi(117 - 48.6 - \left(\frac{-}{15} \right.$$

$$\pi(68.4) - \frac{(-65+3)}{15}$$

$$\pi(68.4 + 12.4) = 80.8\pi$$

cubic

6. Show that (5 points)

a) $\int \sin ax \cos ax \, dx = \frac{1}{2a} \sin^2 ax + c$

b) Evaluate $\int \sin 3x \cos 4x \, dx$

27

28

Part II: Discuss or Define or Explain the following concepts (6marks)

1) The Mean Value Theorem (for Derivatives)

2) The Fundamental Theorem of Calculus

3) The difference between indefinite integral and definite integral

Part III: Work out the following problems clearly (22marks)

1) A man wants to enclose a rectangular field with a fence. He has 500feet of fencing material. And one side of the field will not need to be fenced. Determine the dimensions of the field that will enclose the largest area. (4pts)

2) Sketch the graph of $f(x) = \frac{x}{x+2}$. (4pts)

3) Find an approximation to the integral $\int_1^2 -2x\,dx$ using a Riemann sum with left end points and $n = 5$. (4pts)

4) Evaluate the following integrals. (2pts each)

a) $\int \frac{\sin x}{1-\sin^2 x} dx$

b) $\int \frac{x}{x+2} dx$

c) $\int x^2 e^{-x} dx$

5) Determine the area of the region enclosed by $y = \sqrt{x}$ and $y = x^2$. (4pts)

28